# MATH 1650 FUNCTION INVERSES

**QUESTION:** If $f$ is invertible, what is $(f^{-1})^{-1}$? $(f^{-1})^{-1}$ is the function that 'undoes' $f^{-1}$, so $(f^{-1})^{-1} = f$.

**EXAMPLE:** (Continued.) Let $f(x) = \dfrac{2x-3}{x+1}$ and $g(x) = \dfrac{x+3}{2-x}$.

- Verify $(g \circ f)(x) = x$ for all $x$ in the domain of $f$:

  We first note the domain of $f$ is all real numbers except $x = -1$. So for $x \neq -1$:

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g\left(\frac{2x-3}{x+1}\right) \\
&= \frac{\left(\dfrac{2x-3}{x+1}\right) + 3}{2 - \left(\dfrac{2x-3}{x+1}\right)} \\
&= \frac{\left(\dfrac{2x-3}{x+1}\right) + 3}{2 - \left(\dfrac{2x-3}{x+1}\right)} \cdot \frac{x+1}{x+1} \\
&= \frac{(2x-3) + 3(x+1)}{2(x+1) - (2x-3)} \\
&= \frac{2x - 3 + 3x + 3}{2x + 2 - 2x + 3} \\
&= \frac{5x}{5} \\
(g \circ f)(x) &= x \checkmark
\end{aligned}$$

- Use $g^{-1}(x)$ to solve: $\dfrac{x+3}{2-x} = 1$

  Since $g(x) = \dfrac{x+3}{2-x}$, we may view $\dfrac{x+3}{2-x} = 1$ as $g(x) = 1$.

  Since $g$ is invertible, we get $x = g^{-1}(1)$ and since $g^{-1} = f$, $x = g^{-1}(1) = f(1) = \dfrac{2(1)-3}{(1)+1} = -\dfrac{1}{2}$.

**EXAMPLE:** (Continued.) $f(x) = \dfrac{2x}{x-1}$ and $f^{-1}(x) = \dfrac{x}{x-2}$.

Verify: $\left(f \circ f^{-1}\right)(x) = x$ for all $x$ in the domain of $f^{-1}$: For all $x \neq 2$:

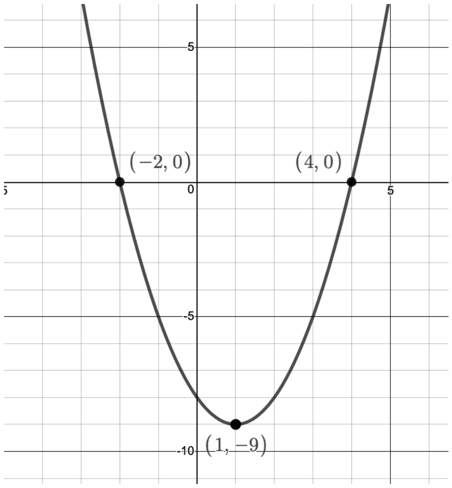
$$
\begin{aligned}
\left(f \circ f^{-1}\right)(x) &= f\left(f^{-1}(x)\right) \\
&= f\left(\frac{x}{x-2}\right) \\
&= \frac{2\left(\dfrac{x}{x-2}\right)}{\left(\dfrac{x}{x-2}\right) - 1} \\
&= \frac{2\left(\dfrac{x}{x-2}\right)}{\left(\dfrac{x}{x-2}\right) - 1} \cdot \frac{x-2}{x-2} \\
&= \frac{2x}{x-(x-2)} \\
&= \frac{2x}{x-x+2} \\
&= \frac{2x}{2} \\
\left(f \circ f^{-1}\right)(x) &= x \checkmark
\end{aligned}
$$

**EXAMPLE:** Consider: $f(x) = \dfrac{2x}{x-1}$ and $f^{-1}(x) = \dfrac{x}{x-2}$

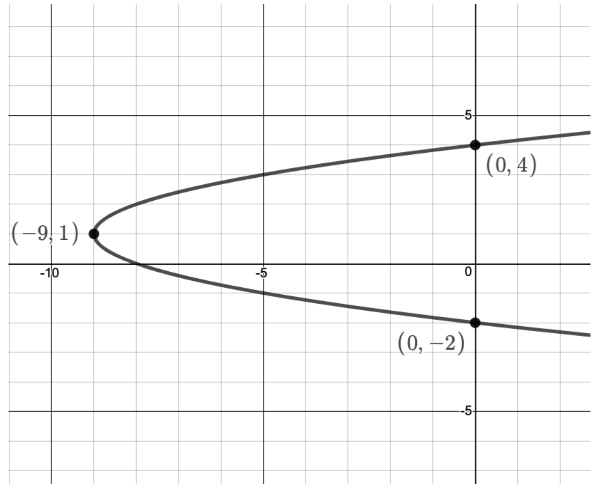
- What is the **vertical** asymptote to the graph of $y = f(x)$? $x = 1$
- What is the **horizontal** asymptote to the graph of $y = f^{-1}(x)$? $y = 1$
- What is the **horizontal** asymptote to the graph of $y = f(x)$? $y = 2$
- What is the **vertical** asymptote to the graph of $y = f^{-1}(x)$? $x = 2$

**NOTE:** The $x$ and $y$ switch meaning the vertical and horizontal asymptotes switch as well!

**EXAMPLE:** Use the graph of $y = f(x)$ on the left to (try to) sketch the graph of $y = f^{-1}(x)$ on the right. What goes wrong?

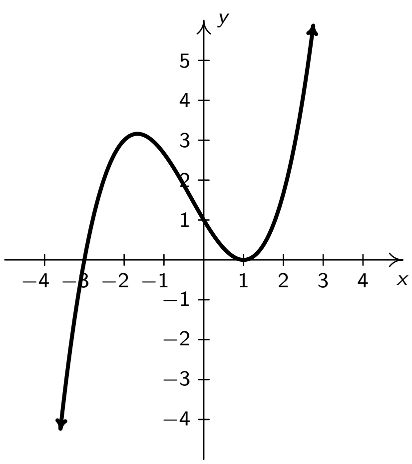



$y = f(x)$

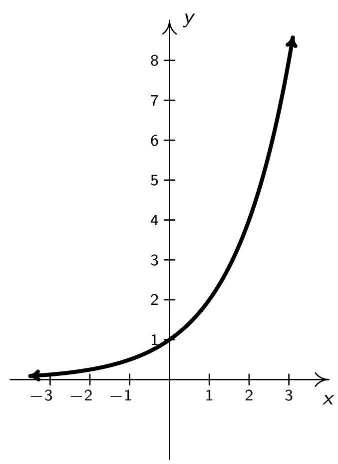



$y = f^{-1}(x)$ - doesn't pass the VLT!

**EXAMPLE:** Use the Horizontal Line Test to determine which of the functions below have inverses.



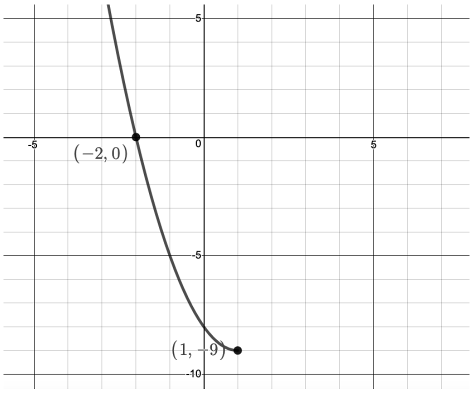
$y = f(x)$ - fails HLT



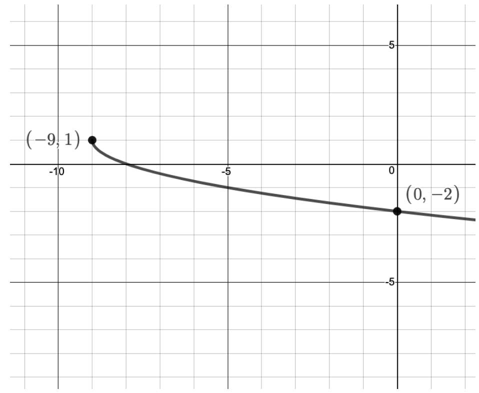
$y = g(x)$ - passes HLT

$f$ does not have an inverse, $g$ does have an inverse.

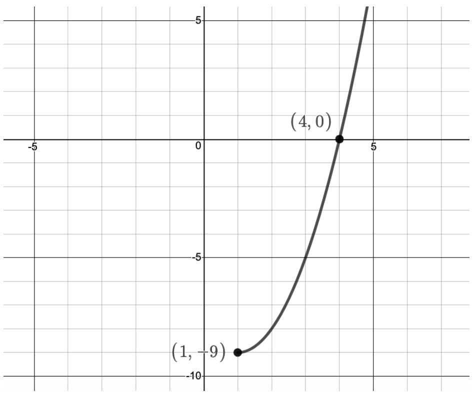
**EXAMPLE:** Graph the inverses of $g$ and $h$ below.



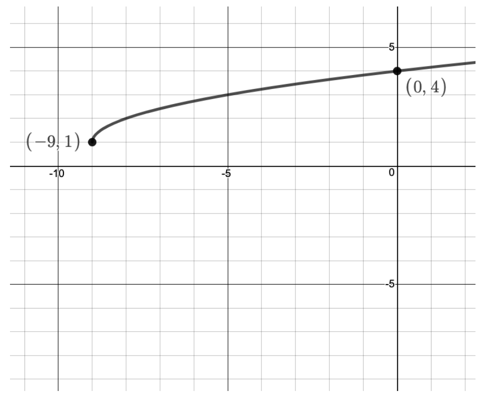

$y = g(x)$




$y = g^{-1}(x)$.




$y = h(x)$




$y = h^{-1}(x)$.

**EXAMPLE:** Let $f(x) = 10x - x^2$ for $x \leq 5$.

- Check your answer algebraically by:
  - simplifying $\left(f \circ f^{-1}\right)(x)$.

$$\begin{aligned}
\left(f \circ f^{-1}\right)(x) &= f\left(5 - \sqrt{25 - x}\right) \\
&= 10\left(5 - \sqrt{25 - x}\right) - \left(5 - \sqrt{25 - x}\right)^2 \\
&= 50 - 10\sqrt{25 - x} - \left(25 - 10\sqrt{25 - x} + (25 - x)\right) \quad \text{FOIL: } \left(5 - \sqrt{25 - x}\right)^2 \\
&= 50 - 10\sqrt{25 - x} - \left(50 - x - 10\sqrt{25 - x}\right) \\
&= 50 - 10\sqrt{25 - x} - 50 + x + 10\sqrt{25 - x} \\
\left(f \circ f^{-1}\right)(x) &= x \checkmark
\end{aligned}$$